Sequences 9.1

A sequence $\{a_n\}$ is an ordered collection of numbers defined by a function f, on a set of sequential integers.

Ex: Fibonacci sequence!

$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_n + F_{n+1}$

$1, 1, 2, 3, 5, 8, \ldots$

terms of the sequence

Find a general term, also known as the nth term of the sequence whose terms are: $1, 6, 11, 16, \ldots$

$n=1, n=2, n=3, n=4 \quad n^{th}?$

Observe: $5n-4$: general form or nth term

Ex: $1,\ x,\ \frac{x^2}{2},\ \frac{x^3}{6},\ \frac{x^4}{24},\ \frac{x^5}{120},\ \ldots$

Find a general formula for the terms of the sequence

Factorial:

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$0! = 1$

General form: $\frac{x^n}{n!}$

$\frac{x^0}{0!},\ \frac{x^1}{1!},\ \frac{x^2}{2!},\ \frac{x^3}{3!},\ \frac{x^4}{4!}\ \ldots\ldots$

Limit of a sequence

If $\lim_{n\to\infty}\{a_n\} = L$, we say that the sequence $\{a_n\}$ converges to L. If the limit does not exist, we say that the sequence diverges.

Ex: Sequence: $3+1,\ 3+\frac{1}{2},\ 3+\frac{1}{3},\ 3+\frac{1}{4},\ \ldots$

Determine whether or not the sequence converges!

Solution: Note: $3+\frac{1}{n}$, $n=1,2,\ldots$

Now $\lim_{N\to\infty}\left(3+\frac{1}{N}\right) = 3+\lim_{N\to\infty}\left(\frac{1}{N}\right)$

$= 3+0 = 3$

Conclusion: $\left\{3+\frac{1}{n}\right\}$, $n=1,2,\ldots$ converges to 3!

9.2 Infinite series

$$\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} + \cdots$$

The exact value of this number is still unknown: series

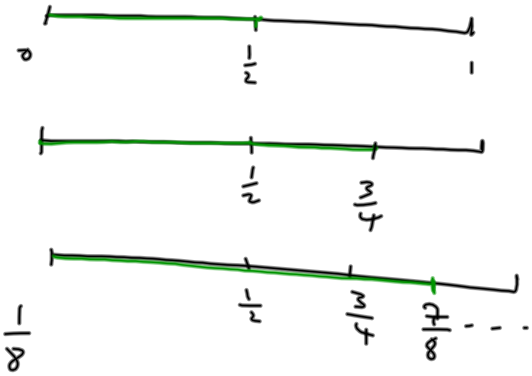
$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

Eg: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Finding infinite sums

Eg:



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

S_1 1st partial sum

S_2

S_3

$\vdots$

S_N nth partial sum

Convergence of an infinite series:

$\sum_{n=0}^{\infty} a_n$ converges to L if the sequence of its partial sum $\{S_N\}$ converges to L. In other words, if $\lim_{N\to\infty} \{S_N\} = L$

then we write $\sum_{n=0}^{\infty} a_n = L$

Eg: Find the sum:

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \cdots$$

One way: Partial Sums:

$$S_1 = \frac{1}{2}$$

$$S_2 = S_1 + \frac{1}{6} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = S_2 + \frac{1}{12} = \frac{4}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = S_3 + \frac{1}{20} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$$

$$\vdots$$

$S_N = \frac{N}{N+1}$ general form of the sequence of partial sums!

Now $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left\{\frac{N}{N+1}\right\} = 1$ (converges!)

Conclusion: $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \cdots = 1$

Observe: $\sum^{\infty} \frac{1}{\quad} = 1$

ALTERNATIVELY:

Observe that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

$$\Longrightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) +$$

$$\left(\cancel{\frac{1}{3}} - \frac{1}{4} \right) + \cdots \left(\frac{1}{N-1} - \cancel{\frac{1}{N}} \right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1} \right) + \cdots$$

$$S_N = 1 - \frac{1}{N+1}$$

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right) = 1 - 0 = \boxed{1}$$

Q.: Show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = 1$

(Telescoping Series)

Sum of geometric series:

Series whose terms are of the form ar^n, $a \neq 0$, $r \neq 0$ are called geometric with ratio r

Eg: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

$\sum_{n=0}^{\infty} 1\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^n}$ geometric series with $a = 1$

ratio $= \frac{1}{2}$

$$\sum_{n=0}^{\infty} ar^n = a + ar^1 + ar^2 + \cdots + ar^N + \cdots$$

$$S_N = a + ar^1 + ar^2 + \cdots ar^N$$

$$rS_N = ar + ar^2 + ar^3 + \cdots + ar^{N+1}$$

$$S_N - rS_N = a + (ar^1 - ar^1) + (ar^2 - ar^2) + \cdots (ar^N - ar^N) + ar^{N+1}$$

$$S_N - rS_N = a + ar^{N+1}$$

$$\Longrightarrow S_N(1-r) = a(1+r^{N+1})$$

$$\Longrightarrow S_N = \frac{a(1+r^{N+1})}{1-r}$$

If $|r|<1$, $\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{a+ar^{N+1}}{1-r}\right)$

$$= \lim_{N\to\infty}\left(\frac{a}{1-r}\right) + \lim_{N\to\infty}\left(\frac{ar^{N+1}}{1-r}\right) \to 0$$

$$\lim_{N\to\infty} S_N = \frac{a}{1-r} \Longrightarrow \boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{when } |r|<1}$$

In general, a geometric series converges when $|r|<1$ and its sum is

$$\boxed{\sum_{n=M}^{\infty} ar^n = \frac{ar^M}{1-r}} \qquad M \geq 0$$

Eg: Evaluate: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$

Observe: $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n \implies r = \frac{1}{5} < 1 \implies$ convergence!

$$S = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right) = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \boxed{\frac{5}{4}} \qquad \frac{a}{1-r}$$

Eg: Evaluate $\sum_{n=2}^{\infty} 4\left(\frac{-3}{4}\right)^n$

$a = 4$

$r = -\frac{3}{4} \to |r|<1$

$$= \frac{4\left(\frac{-3}{4}\right)^2}{1-\left(\frac{-3}{4}\right)}$$

$$= \frac{4\left(\frac{9}{16}\right)}{\frac{7}{4}} = \boxed{\frac{9}{7}}$$

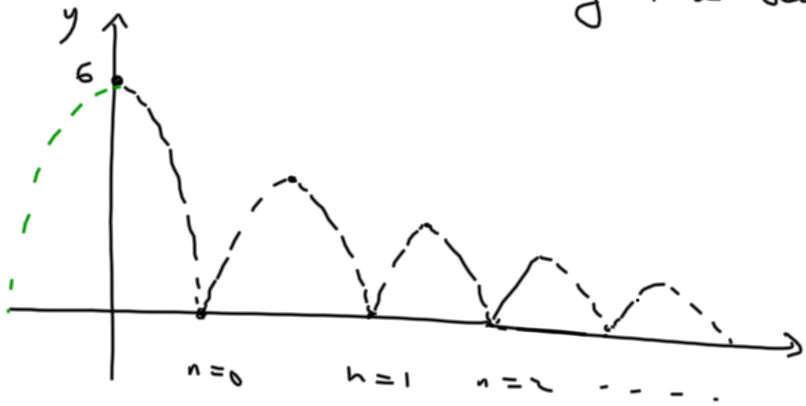
ES: Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2+3^n}{5^n} \right)$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{5^n} + \frac{3^n}{5^n} \right) =$$

$$= \sum_{n=0}^{\infty} 2\left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= \frac{2}{1-\frac{1}{5}} + \frac{1}{1-3/5} = \boxed{5}$$

Eg: A ball is dropped from a height 6ft and begins bouncing. The height of each bounce is $\frac{3}{4}$ of the height of the previous bounce. Find the total vertical distance travelled by the ball.



initial: 6

next distance: $\frac{3}{4}(12)$

next distance: $\left(\frac{3}{4}\right)^2(12)$

⋮

Total distance: $6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \dots$

$$D = \sum_{n=0}^{\infty} 12\left(\frac{3}{4}\right)^n - 6 = +\underline{12} + 12\left(\frac{3}{4}\right) + \dots - 6$$

$$= \frac{12\left(\frac{3}{4}\right)^0}{1-3/4} - 6 = \frac{12}{\frac{1}{4}} - 6 = 48\text{ ft} - 6$$

$$= \boxed{42\text{ ft}}$$

OR $6 + \displaystyle\sum_{n=1}^{\infty} 12\left(\frac{3}{4}\right)^n$